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Conversely, if the seven constants A, B, and c are merely so chosen as to satisfy these three relations and t is then determined by (15), the functions ξ, η, ζ will satisfy (14) and hence x, y, z will satisfy (9).

By multiplying each of the equations (12) by (15), and placing $A_i c = a_i$, $B_i c = b_i$, we have the theorem:

The equations of any geodesic line may be put into the form

$$\left.\begin{aligned} u &= \frac{a_1}{k}\tan ks + b_1, \\ v &= \frac{a_2}{k}\tan ks + b_2, \\ w &= \frac{a_3}{k}\tan ks + b_3, \\ t &= c\sec ks, \end{aligned}\right\} \tag{16}$$

where the constants are connected by the relations

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 = 1 + k^2(b_1^2 + b_2^2 + b_3^2) = c^2, \\ a_1 b_1 + a_2 b_2 + a_3 b_3 = 0\,; \end{aligned}\right\} \tag{17}$$

and, conversely, any equations of the form (16) *where the constants satisfy* (17) *represent a geodesic line, if they are satisfied by any points lying in the space under consideration.*

It is important to know that *to any geodesic line corresponds only one set of the above constants, except that all the signs of a_1, a_2, a_3 and c may be changed, and conversely.* For, a geodesic line is completely determined by a point (x_0, y_0, z_0) on it and a direction (α, β, γ), where the signs of all three of the quantities α, β, γ may be changed without altering the line. Then s_0, the value of s corresponding to (x_0, y_0, z_0), is determined by

$$\tan ks_0 = \alpha \tan kx_0 + \beta \tan ky_0 + \gamma \tan kz_0,$$

which is obtained by differentiating the last of equations (6) and (16). The

value of s_0 is unique within restricted portions of space. The value of c is then determined by

$$c \sec ks_0 = \sec kx_0 \sec ky_0 \sec kz_0,$$

and the values of a_1, a_2, a_3 are found by differentiating the first three of equations (6) and (16). These latter values depend linearly on α, β, γ. Finally b_1, b_2, b are found from (6) and (16). In this way all the constants are uniquely determined. The converse is proved by noticing that if the equations (16) are given and if, in conjunction with (6), they determine a line lying in the space under consideration, the direction of this geodesic is uniquely determined at any one of its points, and therefore the geodesic is unique. No curve at all results from the equations, however, unless at least one point (x_0, y_0, z_0) within the space considered satisfies the equations.

The equations of the geodesic line are readily written in terms of any point upon it. For, let u_0, v_0, w_0 correspond to any such point and let s_0 be the corresponding value of s. These values determine b , b_2, b_3 and the first three of equations (16) become

$$\left.\begin{aligned} u &= \frac{a_1}{k}(\tan ks - \tan ks_0) + u_0, \\ v &= \frac{a_2}{k}(\tan ks - \tan ks_0) + v_0, \\ w &= \frac{a_3}{k}(\tan ks - \tan ks_0) + w_0. \end{aligned}\right\} \tag{19}$$

These are of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3}.$$

Conversely, any equations of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3},$$

when u_0, v_0, w_0 correspond to a point (x_0, y_0, z_0) of space, determine a geodesic line.

For, the equations may be written

$$\frac{u - (u_0 + \lambda c_1)}{\rho c_1} = \frac{v - (v_0 + \lambda c_2)}{\rho c_2} = \frac{w - (w_0 + \lambda c_3)}{\rho c_3},$$

where λ and ρ are undetermined multipliers. If we place

$$a_1 = \rho c_1, \qquad b_1 = u_0 + \lambda c_1,$$
$$a_2 = \rho c_2, \qquad b_2 = v_0 + \lambda c_2,$$
$$a_3 = \rho c_3, \qquad b_3 = w_0 + \lambda c_3,$$

determine λ and ρ by (17) and substitute the values of a, b thus obtained in (16), we see that the geodesic line thus defined coincides with the given locus.

To obtain the equation of the geodesic surface, consider a pencil of geodesic lines with its vertex at (u_0, v_0, w_0). Since the constants c_1, c_2, c_3 of the last theorem are connected with the coordinates α, β, γ of the direction of the geodesic line by linear relations, the pencil has the equations

$$\frac{u - u_0}{\lambda c_1' + \mu c_1''} = \frac{v - v_0}{\lambda c_2' + \mu c_2''} = \frac{w - w_0}{\lambda c_3' + \mu c_3''},$$

where c_1', c_1'' etc. are constants. The elimination of λ and μ gives a linear equation

$$lu + mv + nw + p = 0.$$

Conversely, let any such linear equation be given, which is satisfied by at least one point (u_0, v_0, w_0) of the space considered. The necessary and sufficient conditions that a geodesic line

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3}$$

should lie on the corresponding surface are that

$$lu_0 + mv_0 + nw_0 + p = 0,$$
$$lc_1 + mc_2 + nc_3 = 0.$$

The first condition means that the point (u_0, v_0, w_0) lies on the surface. The second is satisfied by

$$c_1 = \lambda c_1' + \mu c_1'', \quad c_2 = \lambda c_2' + \mu c_2'', \quad c_3 = \lambda c_3' + \mu c_3'',$$

where c_1', c_2', c_3' and c_1'', c_2'', c_3'' are any two sets of values which satisfy it, and λ and μ are variable parameters. From any point of the surface, therefore, radiates a pencil of geodesic lines lying wholly on the surface. Moreover, it is easy to see that any point of the surface within a restricted region about the vertex of the pencil is reached by some line of the pencil. To sum up:

Any geodesic surface is represented by a linear equation in u, v, w; and, conversely, any linear equation in u, v, w represents a geodesic surface, if it represents anything at all in space.

Any point of a geodesic surface may be taken as the vertex of a generating pencil of geodesic lines.

The forms of the equations of geodesic lines and surfaces are the same as those of straight lines and planes in Euclidean geometry. They have, therefore, in common with straight lines and planes those properties which may be deduced from the forms of the equations. In particular we note the following:

THEOREM 8. *Any three points not in the same geodesic line and lying in a restricted portion of space determine a geodesic surface.*

THEOREM 9. *If two points on a geodesic surface are connected by a geodesic line, the line lies wholly on the surface.*

THEOREM 10. *Any two geodesic surfaces intersect in a geodesic line if they intersect at all.*

7. Displacements. We shall now show that our space satisfies the third hypothesis.

For that purpose it will be convenient to introduce an auxiliary space, which shall be Euclidean in its properties and in which u, v, and w shall be the rectangular Cartesian coordinates of a point. This auxiliary space we shall denote by S', while the space of constant curvature, whose properties we are investigating, we shall call S. The quantities u, v, w have then a double interpretation; on the one hand, in S, they are defined by (6), on the other hand in S' they are ordinary Cartesian coordinates. We thus establish a point for point correspondence between S and S' by which geodesic lines and surfaces in S correspond respectively to straight lines and planes in S'.

We proceed now to ask what corresponds in S' to the geodesic distance between two points in S. For that purpose consider the equation of a geodesic line in the form (16) and let A and B be two points upon it, to which correspond respectively the values s_1, u_1, v_1, w_1 and s_2, u_2, v_2, w_2. Then the geodesic distance between A and B is $s_2 - s_1$. From the equations of the geodesic line, with regard to the relations (17) which connect the constants, we readily compute

$$k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1 = c^2(\tan ks_1 \tan ks_2 + 1)$$

$$k^2(u_1^2 + v_1^2 + w_1^2) + 1 = c^2 \sec^2 ks_1,$$

$$k^2(u_2^2 + v_2^2 + w_2^2) + 1 = c^2 \sec^2 ks_2,$$

whence follows

$$\cos(s_2 - s_1) = \frac{k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}.$$

This formula which we have obtained in S has a good significance in S'. For let us take in S' the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

as the fundamental quadric in a Cayley system of projective measurement. Then, if M is the distance between two points (u_1, v_1, w_1) and (u_2, v_2, w_2), we have, in accordance with a well known formula of projective geometry,

$$\cos M = \frac{k^2(u\ u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}$$

and hence

$$M = s_2 - s_1.$$

Hence we may say: *If we adopt in S' a system of projective measurement referred to the quadric surface*

$$k^2(u^2 + v^2 + w^2) + 1 = 0,$$

then the geodesic distance between two points in S is equal to the projective distance between the two corresponding points in S'.

Now in order that we may obtain a displacement in S it is necessary and sufficient that we find a continuous one-to-one correspondence between the points of S by means of which geodesic lines go into geodesic lines and the length of any portion of a geodesic line is unchanged. That these conditions are necessary, was shown in §4; that they are sufficient may be shown by considering the line element as the length of a geodesic line between two indefinitely near points (x, y, z) and $(x + dx, y + dy, z + dz)$.

To any displacement in S corresponds accordingly a transformation in S' by which straight lines go into straight lines and the projective length of any portion of a straight line is unchanged.

The projective geometry teaches that the transformations with these properties are the collineations of which the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant. Accordingly we may say:

The displacements in S correspond to the collineations in S' of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant; and conversely.

These collineations are readily written down. They are

$$\left.\begin{aligned} u' &= \frac{\alpha_1 u + \alpha_2 v + \alpha_3 w + \alpha_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ v' &= \frac{\beta_1 u + \beta_2 v + \beta_3 w + \beta_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ w' &= \frac{\gamma_1 u + \gamma_2 v + \gamma_3 w + \gamma_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \end{aligned}\right\} \tag{18}$$

where $|\alpha_1 \beta_2 \gamma_3 \delta_4| \neq 0$, and the sixteen constants are connected by the ten conditions:

$$\left.\begin{aligned} k^2(\alpha_i^2 + \beta_i^2 + \gamma_i^2) + \delta_i^2 &= k^2, \qquad (i = 1, 2, 3) \\ k^2(\alpha_4^2 + \beta_4^2 + \gamma_4^2) + \delta_4^2 &= 1, \\ \alpha_i \alpha_h + \beta_i \beta_h + \gamma_i \gamma_h + \delta_i \delta_h &= 0. \qquad (i, h = 1, 2, 3, 4 : i \neq h) \end{aligned}\right\} \tag{19}$$

We now need to show that the displacements (18) satisfy the demands of the third hypothesis. To prove this directly we might show first that the angle between two geodesic lines in S is equal to the angle between the corresponding lines in S', provided the latter angle is measured with reference to the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0.$$

We might then show directly that the third hypothesis is satisfied in S', and hence in S. But this method would demand extended reckoning. We shall choose an indirect method and shall show first that the third hypothesis is satisfied in a particular point, the origin, and shall then show that it is consequently satisfied in any point of space.

Consider then the origin. The origin is unique in our system of coordinates because there, and there only, the angle between two intersecting geo-

desic lines in S is equal to the Euclidean angle between the two corresponding straight lines in S'.

To show this, consider a geodesic line in the form (16). If this line goes through the origin with the direction (ξ_1, ξ_2, ξ_3), we have by applying the methods of p. 93,

$$s_0 = 0, \quad b_1 = b_2 = b_3 = 0,$$
$$a_1 = \xi_1, \quad a_2 = \xi_2, \quad a_3 = \xi_3, \quad c = \pm 1.$$

The equation of §2 for the angle between two such lines becomes

$$\begin{aligned}\cos\theta &= \xi_1\,\xi_1' + \xi_2\,\xi_2' + \xi_3\,\xi_3' \\ &= a_1\,a_1' + a_2\,a_2' + a_3\,a_3'.\end{aligned}$$

The corresponding straight lines in S' are

$$\frac{u}{a_1} = \frac{v}{a_2} = \frac{w}{a_3}$$

and

$$\frac{u}{a_1'} = \frac{v}{a_2'} = \frac{w}{a_3'},$$

whence, if ϕ is the Euclidean angle between them,

$$\begin{aligned}\cos\phi &= a_1\,a_1' + a_2\,a_2' + a_3\,a_3' \\ &= \cos\theta.\end{aligned}$$

With proper conventions as to corresponding angles, we have

$$\phi = \theta.$$

Consider now a displacement in S for which the origin is a fixed point. From (18) and (19), it will correspond in S' to a collineation of the type

$$\left.\begin{aligned} u' &= \alpha_1 u + \alpha_2 v + \alpha_3 w, \\ v' &= \beta_1 u + \beta_2 v + \beta_3 w, \\ w' &= \gamma_1 u + \gamma_2 v + \gamma_3 w, \end{aligned}\right\} \quad (R)$$

where

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1, \qquad (i = 1, 2, 3)$$
$$\alpha_i\,\alpha_h + \beta_i\,\beta_h + \gamma_i\,\gamma_h = 0. \qquad (i, h = 1, 2, 3;\ i \neq h)$$

This is a Euclidean rotation about the origin, or a rotation combined with a reflection on one of the coordinate planes, according as $|\alpha_1\,\beta_2\,\gamma_3| = \pm 1$. We shall use the rotation only and shall denote it by (R). Let now OK and

OL be any two geodesic lines in S intersecting at the origin and let OM and ON be any two other lines, such that the angle KOL equals the angle MON. We wish to show that there exists a displacement by which OK becomes OM and OL becomes ON.

In S' we have four corresponding straight lines $O'K'$, $O'L'$, $O'M'$, $O'N'$ such that the Euclidean angle $K'O'L'$ equals the angle $M'O'N'$. Now there exists in S' a rotation (R) by which $O'K'$ may be brought into coincidence with $O'M'$ and $O'L'$ with $O'N'$. This is a well known fact which of course has its analytic proof. Hence we conclude that the required displacement exists in S.

The third hypothesis is accordingly satisfied at the origin.

We shall now show that the hypothesis is satisfied at any point of space. Let P be any point of S and PK, PL, PM, PN four geodesic lines such that the angle KPL equals the angle MPN. We wish to show that there exists a transformation of type (18) by which PK goes into PM and PL into PN. Among the transformations (18) there are ∞^3 transformations by means of which P goes into O. Take any particular one of these, no matter which, and call it T. Let PK, PL, PM and PN become OK_1, OL_1, OM_1 and ON_1 by means of T. Then the angle K_1OL_1 equals the angle M_1ON_1, since all displacements preserve angles. Then there exists, as we have seen, a displacement R_1 by which OK_1 goes into OM_1, and OL_1 into ON_1. Let T^{-1} be the inverse transformation of T. Then, if we carry out in order the transformations T, R_1, and T^{-1}, we have a transformation which transforms PK into PM and PL into PN. Moreover, this transformation is of type (18). This follows geometrically from the fact that all the transformations we have used are collineations of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is invariant, and (18) is the general form of such a collineation.

Hence *the third hypothesis is satisfied in any space of constant curvature.*

We are accordingly justified in proving general propositions by taking the lines involved in any convenient position. We may, for example, apply to any geodesic line, a result already proved for the line OX and may state the following theorem:

THEOREM 11. *Within a properly restricted portion of a geodesic surface, one and only one geodesic line can be drawn perpendicular to a given geodesic line from a point outside of it.*

Corollary. Within a properly restricted portion of space, one and only one perpendicular can be drawn to a geodesic surface from a point outside of it.

8. Space of Zero Curvature. If $k = 0$, Riemann's form of the line element is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The coordinates of the direction (ξ_1, ξ_2, ξ_3) of a geodesic line are connected by the relation

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

and the angle between two geodesic lines is given by the formula

$$\cos\theta = \xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3'.$$

The differential equations of the geodesic lines are

$$\frac{d^2x}{ds^2} = 0, \qquad \frac{d^2y}{ds^2} = 0, \qquad \frac{d^2z}{ds^2} = 0.$$

The general equations of the geodesic lines are, therefore,

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3} = s$$

where $$a_1^2 + a_2^2 + a_3^2 = 1.$$

The equation of the geodesic surfaces is

$$lx + my + nz + p = 0.$$

It is evident that the coordinates x, y and z are equivalent to u, v and w of the previous paragraphs, and that Theorems 1–11 still hold. In addition we may deduce a theorem relating to the angles of a triangle which is peculiar to a space of zero curvature. Before stating the theorem, however, we need to make the conceptions of a triangle and of the angles of a triangle precise.

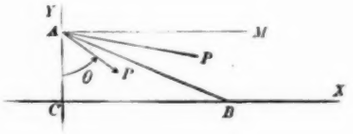
If we select, in the region in which Theorems 1–11 hold, any three points not in the same geodesic line, we may connect them in pairs by geodesic lines and pass a geodesic surface through them. The geodesic lines lie then on this geodesic surface, and the portion of the latter bounded by them we call a *triangle*. The triangle divides the points of the surface not lying on its sides into two classes, *interior* and *exterior* points of the triangle.

To remove the ambiguity inherent in the determination of an angle from the formula above given, we may make the following convention as to the angles of a triangle. If the vertices of the triangle are A, B, and C, the angle at A shall be determined by considering the geodesic lines which radiate from A and enter the triangle ABC. If now a variable line is considered to start from coincidence with AB and take successively the positions of all these lines until it coincides with AC, the angle θ made by the varying line with AB varies continuously from zero to a final value A, which we call the *angle* A of the triangle. We may now state the theorem:

THEOREM 12. *Within a properly restricted region of a space of zero curvature, the sum of the angles of any triangle equals* π.

It will be sufficient to prove this theorem for right triangles, for any triangle may be divided into two right triangles by dropping a perpendicular from any vertex to the opposite side, and if the theorem is true for the two right triangles thus formed, it is evidently true for the original triangle.

Consider the geodesic surface $z = 0$, and a point A of which the (xyz)-coordinates are $(0, p, 0)$, where p is positive. Let A and $C(0, 0, 0)$ determine a geodesic line, on which the direction AC is that of the increasing s. Then the direction of AC is $(a_1 = 0,\ a_2 = -1,\ a_3 = 0)$.



Consider now a moving geodesic turning about A continuously in one direction, thereby fixing the angle θ between AC and AP. By Theorem 1, this line will meet CX at the beginning of its motion. For convenience we take the direction of motion to be such that AP meets CX on the positive side of the origin. Let AM be the first position in which the moving geodesic is perpendicular to AC. If the direction of the increasing s is always taken from A, the direction of AM is $(\beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = 0)$. Then the direction of AP is

$$\xi_1 = a_1 \cos\theta + \beta_1 \sin\theta = \sin\theta,$$
$$\xi_2 = a_2 \cos\theta + \beta_2 \sin\theta = -\cos\theta,$$
$$\xi_3 = a_3 \cos\theta + \beta_3 \sin\theta = 0,$$

and hence the equations of AP are

$$x = s \sin\theta,$$
$$y = -s\cos\theta + p,$$
$$z = 0.$$

Let now AB be a line AP which meets CX, and let B be $(0, q, 0)$, where, by the conventions made, q is positive. If we denote by A the angle CAB, we have from the equations of the line

$$\tan A = \frac{q}{p}.$$

Similarly we may show that

$$\tan B = \frac{p}{q},$$

and hence

$$\cot B = \tan A.$$

Since p and q are both positive, and A and B are both acute angles,

$$B = \frac{\pi}{2} - A.$$

The angle $C = \frac{\pi}{2}$, and hence we have $A + B + C = \pi$, as was to be proved.

As has been repeatedly said, the theorems so far obtained have been shown to be valid only in a region in which there exists a one-to-one relation between the points of space and the coordinates (x, y, z). It is, however, possible to demand that this relation exist for the entire space and all possible values of the coordinates. In this case we have

Euclidean Space.

Here all the theorems so far deduced hold without restriction, and we have in addition two others of interest.

Theorem 13. *In Euclidean space every geodesic line is of infinite length.*

This follows at once from the form of the equations.

Theorem 14. *On any geodesic surface in Euclidean space, one and only one geodesic line can be drawn through a given point parallel to a given geodesic line; any other geodesic line through the point intersects the given geodesic.*

We define *parallel lines* as follows: A geodesic line through a given point is said to be parallel to a given geodesic line, when it is the limit ap-

proached by a geodesic line through the given point intersecting the given geodesic, as the point of intersection becomes indefinitely remote.

The truth of the theorem then follows from the equations used in proving Theorem (12). We had there

$$\tan A = \frac{q}{p}.$$

As the point B recedes indefinitely in either direction, the angle A approaches $\frac{\pi}{2}$, and the parallel line is AM. Any other value of A determines q when p is fixed.

These fourteen theorems form the groundwork of Euclidean geometry.

Euclidean space is not however the only conceivable space of zero curvature. A systematic investigation of all possible cases seems to be yet lacking, although the general lines on which such an investigation would proceed have been marked out by Klein and some of the problems have been attacked by Killing.* We will content ourselves with mentioning three possibilities.

First, we may suppose that to all points with coordinates $(x + n_1 c_1, y + n_2 c_2, z + n_3 c_3)$ where c_1, c_2, c_3 are constants, and n_1, n_2, n_3 are any positive or negative integers, shall correspond the same point of space. Such a conception presents no difficulty, at least analytically. Our space is then represented point for point by the interior of a rectangular parallelopiped in Euclidean space, the lengths of the edges of the parallelopiped being c_1, c_2 and c_3 respectively and the opposite faces corresponding point for point with one another. For any portion of space lying within this parallelopiped, Theorems 1–12 hold, but Theorems 13 and 14 are not always true. For, the lines $(x = \text{const.}, y = \text{const.})$, $(y = \text{const.}, z = \text{const.})$ and $(z = \text{const.}, x = \text{const.})$ are certainly finite in length, and for these lines the theorem of parallels ceases to hold. Moreover triangles may be constructed within the entire space thus defined for which Theorem 12 is not true.

Two other examples of spaces of zero curvature may be derived from the preceding by considering that one or two of the edges of the parallelopiped are indefinite in extent.

9. Space of Constant Negative Curvature. If $k = i/R$ where R is a real quantity, the space is one of constant negative curvature, $-1/R^2$.

* Klein, *Math. Annalen*, vol. 37 (1890) p. 544.
Killing, *ibid.* vol. 39 (1891) p. 257, and *Grundlagen der Geometrie*, vol. 1, chap. 4.

The element of arc may be written

$$ds^2 = \cosh^2\frac{z}{R}\cosh^2\frac{y}{R}\,dx^2 + \cosh^2\frac{z}{R}\,dy^2 + dz^2,$$

and the functions u, v, w, t are

$$u = R\tanh\frac{x}{R},$$

$$v = R\,\mathrm{sech}\,\frac{x}{R}\tanh\frac{y}{R},$$

$$w = R\,\mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\tanh\frac{z}{R},$$

$$t = \mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\,\mathrm{sech}\,\frac{z}{R}.$$

To any set of real values of (x, y, z) correspond then one and only one real set of values of u, v, w, t. Since, however, sech θ is numerically not greater than unity for real values of θ, t is numerically less than unity. That is

$$0 \leqq t^2 \leqq 1$$

and consequently

$$0 \leqq u^2 + v^2 + w^2 \leqq R^2$$

for real values of x, y, and z. Conversely, it is readily proved that to any real values of u, v, w satisfying the above conditions correspond one and only one set of values of x, y, and z.

The angles of a triangle obey the following theorem:

THEOREM 12′. *In a properly restricted portion of a space of constant negative curvature, the sum of the angles of any triangle is less than* π.

We may employ the construction and figure of the previous paragraph, but the analytic work will be different. The direction of the line AP must satisfy the relations,

$$\xi_1^2\cosh^2\frac{p}{R} + \xi_2^2 = 1, \qquad \xi_3 = 0.$$

In particular, the direction of AC is $(a_1 = 0,\ a_2 = -1,\ a_3 = 0)$ and of AM $(\beta_1 = \mathrm{sech}\,\frac{p}{R},\ \beta_2 = 0,\ \beta_3 = 0)$. Hence

$$\xi_1 = \sin\theta\,\mathrm{sech}\,\frac{p}{R}, \quad \xi_2 = -\cos\theta, \quad \xi_3 = 0.$$

By (19), §6 the equations of AP are

$$\left.\begin{aligned} u &= R\tanh\frac{x}{R} = a_1 R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right], \\ v &= R\,\mathrm{sech}\,\frac{x}{R}\tanh\frac{y}{R} = a_2 R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right] + R\tanh\frac{p}{R}, \\ z &= 0, \end{aligned}\right\}$$

where s_0 is the value of s for the point A. From (16), §6,

$$\mathrm{sech}\,\frac{p}{R} = c\,\mathrm{sech}\,\frac{s_0}{R}.$$

Hence, by differentiating the equations of AP and substituting the coordinates of A,

$$\xi_1 = a_1\,\mathrm{sech}^2\frac{s_0}{R} = \frac{a_1}{c^2}\,\mathrm{sech}^2\frac{p}{R},$$

$$\mathrm{sech}^2\frac{p}{R}\,\xi_2 = a_2\,\mathrm{sech}^2\frac{s_0}{R} = \frac{a_2}{c^2}\,\mathrm{sech}^2\frac{p}{R}.$$

Therefore

$$a_1 = c^2\sin\theta\cosh\frac{p}{R},$$

$$a_2 = -c^2\cos\theta,$$

and the equations of AP may therefore be written

$$u = -\tan\theta\cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right).$$

The line AB is satisfied by placing $x = q$, $y = 0$, $\theta = A$; whence

$$\tan A = \frac{\tanh\frac{q}{R}}{\sinh\frac{p}{R}}.$$

Similarly

$$\tan B = \frac{\tanh\frac{p}{R}}{\sinh\frac{q}{R}},$$

and hence

$$\tan A\tan B = \frac{1}{\cosh\frac{p}{R}\cosh\frac{q}{R}}.$$

Since p and q are both positive, $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$, and $\tan A \tan B < 1$; hence

$$A + B < \frac{\pi}{2}$$

and

$$A + B + C < \pi.$$

The theorem therefore holds for right triangles and consequently for all triangles.

We obtain the simplest example of space of constant negative curvature in assuming that a one-to-one correspondence exists between our entire space and the (x, y, z) coordinates. We have then the

SPACE OF LOBACHEVSKY, BOLYAI AND GAUSS.

Within this space, Theorems 1–11 hold universally. In addition we have the following theorems:

THEOREM 13′. *All geodesic lines are infinite in length.*

This follows at once from the equations.

THEOREM 14′. *On any geodesic surface, two geodesic lines can be drawn through a given point parallel to a geodesic line; all other geodesic lines through the given point are separated by the two parallel lines into two sets, one set consisting of lines which intersect the given geodesic line, the other of lines which do not.*

The definition of parallels remains the same as in Euclidean space. The truth of the theorem follows from the equation

$$u = -\tan\theta \cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right)$$

deduced above for a line through $(x = 0, y = p, z = 0)$ in the plane $z = 0$. This line intersects OX, or $v = 0$, in the point

$$u = R\sinh\frac{p}{R}\tan\theta, \qquad v = 0, \qquad w = 0.$$

These values correspond to real, infinite, or imaginary values of x, y, z according as

$$R^2\sinh^2\frac{p}{R}\tan^2\theta < , = , \text{ or } > R^2.$$

Let a be the smallest value of θ for which

$$R^2 \sinh^2 \frac{p}{R} \tan^2 \theta = R^2 ;$$

then
$$\cot a = \sinh \frac{p}{R} = \frac{e^{\frac{p}{R}} - e^{-\frac{p}{R}}}{2}$$

whence
$$e^{\frac{p}{R}} = \cot \tfrac{1}{2} a ,$$

$$a = 2 \cot^{-1} e^{\frac{p}{R}}$$

It is now clear that if $0 \leqq \theta < a$, the geodesic line intersects the line OX, the point of intersection receding indefinitely as θ approaches a; if $a < \theta < \pi - a$, the geodesic line does not intersect OX; if $\pi - a < \theta \leqq \pi$, the geodesic again intersects OX. This establishes the theorem.

We have thus the groundwork of Lobachevsky's geometry.*

Klein† has pointed out that other examples of space of constant curvature may be found by considering the polyedrons used by Poincaré in his investigations of the Klein groups. The details of the work are yet to be carried out.

10. Space of Constant Positive Curvature. If $k = 1/R$, where R is a real quantity, the space is one of constant positive curvature $1/R^2$. The functions u, v, w, t are then

$$u = R \tan \frac{x}{R} ,$$

$$v = R \sec \frac{x}{R} \tan \frac{y}{R} ,$$

$$w = R \sec \frac{x}{R} \sec \frac{y}{R} \tan \frac{z}{R} ,$$

$$t = \sec \frac{x}{R} \sec \frac{y}{R} \sec \frac{z}{R} .$$

It appears that u, v, w, t are single valued functions of x, y, z, but the converse is not true.

If we limit ourselves to such a portion of space that a one-to-one relation exists between its points and the coordinates (u, v, w) and (x, y, z), Theorems 1–11 of the preceding paragraphs hold. In addition we have the following theorem:

* *Cf. Zwei geometrische Abhandlungen*, German translation by Engel, Leipzig, 1899.

† *Math. Annalen*, vol. 37 (1890), p. 544.

THEOREM 12″. *In a properly restricted portion of a space of constant positive curvature, the sum of the angles of any triangle is greater than* π.

The proof runs exactly parallel with the proof of the corresponding theorem in space of constant negative curvature.

We obtain an example of a space of constant positive curvature, if we assume that a one-to-one correspondence exists between the points of space and all the triples of values (u, v, w). We have then

SIMPLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space, the following distinctive theorems are derived from the equations of the geodesic lines:

Every geodesic line is of constant finite length πR.

All geodesic lines intersecting in a point return again to their point of intersection, and have besides no point in common.

There are no parallel lines in this space.

Another example of a space of constant curvature is obtained if we assume that a one-to-one correspondence exists between the points of space and the sets of four values (u, v, w, t) where

$$1 + R^{-2}(u^2 + v^2 + w^2) = t^2.$$

We have then the

DOUBLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space the distinctive theorems hold:

Every geodesic line is of constant finite length $2\pi R$.

All geodesic lines intersecting in a point intersect again in a point at a distance πR *from the point of intersection and then return to the point of intersection.*

Parallel lines do not exist in this space either.

In the double-space the geometry on the geodesic surfaces is identical with that on the surface of a sphere in Euclidean space. In the simple space the geometry on a geodesic surface is that which would be obtained in Euclidean space by the projection of the surface of the sphere from its centre upon a plane.

For further discussion, the reader is referred to the last cited work of Killing.

11. Relation of our Results to the Facts of Experience. There remains yet the problem to enquire how far the foregoing systems of geometry explain the facts of experience. To deal with this question exhaustively would require more space than is at our disposal. We must content ourselves with a mere outline of the treatment.

We do not wish to discuss the metaphysical questions involved. To some minds the space conceptions which underlie Euclid's work, including his theories of parallels and of the infinite extension of space, are *a priori* true, and no others conceivable. With those who hold this opinion we have no argument for we meet them on no common ground. We take the standpoint that our knowledge of external space comes to us solely from experience. A system of geometry is an attempt to explain certain facts of experience and, as long as the system is self-consistent, its objective truth or falsity can be judged only by its success in so doing.

We assert: *Any one of the systems of geometry derived from the hypotheses of this article is, as far as our present knowledge goes, in full accord with all facts of experience, provided suitable values are given to the constants involved in each system.*

Experience is necessarily limited to a finite extent of space. Observation and measurement can never extend beyond the most remote astronomical objects, and this distance can well be very small compared with the full extent of objective space. Hence any geometry which agrees with experience only for a restricted portion of space will serve quite as well as Euclid's geometry for explaining the facts of experience. In fact we are not justified in going outside of such a restricted portion of space. Any assumptions, for example, as to the existence or the nature of parallel lines, the connectivity of space, or the displacement of space as a whole, are only justified in case they produce results which would be visible in an arbitrarily small region of space.

The geodesic lines of any space of constant curvature have for a suitably restricted portion of their length all the properties of the "straight lines" of practical life or of Euclidean geometry. In the endeavor to construct a material line which shall be as nearly as possible "straight," we may proceed by attempting to realize the shortest distance between two points; for example, by free-hand drawing, by stretching a string, or by observing the path of a ray of light. The result is simply a geodesic line by definition. Or, we may look for a line which may be revolved upon itself when two points are fixed. This is also a property of a geodesic line in a space of constant curvature, by virtue of the third hypothesis.

A geodesic surface has the properties of a "plane." The practical testing of a plane surface by the application of a "straight" edge has its full significance in a space of constant curvature.

The practical measurement of length and angle gives no criterion as to the curvature of space. Such measurement consists of the assumption of a unit and the application of the unit to the thing measured. These operations are carried out without hindrance in any space of constant curvature. In fact the difference between a space of constant curvature and Euclidean space is not in the manner in which measurement is practically carried out but in the analytic interpretation of the process. So, more generally, any geometric figure may be carried without distortion from one portion of space of constant curvature to another, so that all properties of figures are independent of the positions of the figures in space.

From these considerations it appears that the groundwork of experimental geometry is unaltered by an assumption as to the curvature of space, provided the curvature is constant. Referring to the theorems of the preceding pages, we see that all spaces of constant curvature agree in the first eleven theorems. A distinction appears first in the twelfth theorem. Here we have apparently a means at hand to determine precisely the curvature of external space by measuring the angles of a triangle. The test however fails, owing to the practical impossibility of exact measurements. All we can discover is that the sum of the angles of a triangle does not differ very much from π. And now it is possible to show that if the sides of a triangle in a space of constant curvature, $\pm 1/R^2$, are sufficiently small compared with R the divergence of the sum of its angles from π is within the limits of the errors of observation.* We can say only this: *If the space of experience is of constant curvature* $\pm 1/R^2$, *R must be very large compared with the distance between any two points lying in that part of space in which measurement is possible.*

To complete our argument, we need to ask whether other crucial theorems may not be developed, by means of which the curvature of space may be practically determined. At present, no such theorem has been found. Euclidean geometry in finite space may be derived from our first twelve theorems, and the geometry in a space of constant curvature agrees, as far as has been shown, with Euclidean geometry within the limits of error of observation.

There are still questions to be solved, however, notably in regard to the

* See for example the calculation in Lobachevsky's *Zwei geometrische Abhandlungen*, translated by F. Engel, pp. 22–24.

connectivity of different spaces of constant curvature. It is only within the last twenty-five years that the possibility has been recognized of the existence of such spaces as that represented by the parallelopiped of §8, and of similar forms with other curvature. Consequently many theorems stated in the earlier papers as universally true are now known to be false, or true only in a restricted portion of space. For example, a space or a surface of zero curvature is not necessarily infinite in extent; two "straight" lines in a space of zero curvature may meet in more than one point; and the sum of the angles of a triangle in a space of zero curvature may exceed π, if the surface of the triangle covers a sufficiently large area. It is possible that a more careful study of these spaces may yet reveal properties which render them unfit to explain the facts of experience, but at present no such properties have been found.

If we are satisfied that the geometries belonging to spaces of constant curvature explain experience, the question naturally arises: *Are these the only systems of geometry which do so?* An answer to this question may be attempted by building up synthetically systems of geometry after the manner of Euclid, but with the assumption of the fewest possible axioms. The most recent example of this mode of procedure is found in the elegant work of Hilbert on the Foundations of Geometry.*

If we maintain the analytic methods of this article, the question demands an examination of our three hypotheses. They have been shown to be sufficient to explain experience; the question now is as to their necessity. As to the first hypothesis it is evident that it must be made, if space is to be treated by the ordinary methods of analysis. With the second hypothesis the case is not so clear, since we have assumed the line element arbitrarily. Helmholtz, however, has obtained this line element as a consequence of certain axioms of motion, which include our third hypothesis. Lie† has shown that Helmholtz's axioms and methods need a thorough revision, but his results remain essentially unaltered. We are, accordingly, justified in saying that no explanation of spatial phenomena has yet been given without the assumption of axioms which lead necessarily to our three hypotheses.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, DECEMBER 1901.

* *Grundlagen der Geometrie*, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Teubner, 1899.

† *Cf.* Lie-Engel: *Theorie der Transformationsgruppen*, vol. 3, pp. 437–543. See also Klein, *Lectures on Mathematics*, *l. c.*

BRILLIANT POINTS AND LOCI OF BRILLIANT POINTS.*

By W. H. Roever.

1. Introduction. When a ray of light meets a polished surface it is reflected according to the following laws:

The incident ray, the reflected ray, and the normal to the reflecting surface at the point of reflection lie in one plane. This plane is called the plane of reflection.

In the plane of reflection, the incident ray and the reflected ray lie on opposite sides of the normal and on the same side of the tangent.

The incident ray and the reflected ray make equal angles with the normal.

These laws are inferred from experiments in which pencils of light are reflected from polished planes.

The points of reflection of those reflected rays which pass through the pupil of an observer's eye are, for the observer, luminous points, and are called *brilliant points.*† The points of reflection of those reflected rays which, when produced backward, pierce the pupil of an observer's eye are *not* luminous, and are sometimes called *virtual* brilliant points. Virtual brilliant points present themselves in the analytic treatment when certain equations are freed from radicals.

Tubular surfaces are envelopes of spheres of constant radii whose centres are situated in a given space curve which is called the axis of the tube. As the radius of a tubular surface approaches zero, its brilliant points approach points in the axis. The axis is the given space curve, and this fact suggests the definition of a brilliant point of a space curve.

It is evident that the position of a brilliant point depends upon the position of the observer's eye, the position and nature of the source of light, and the position of the reflecting surface. Hence when the surface, source or eye moves, or when the surface changes its shape or size, the brilliant points move. The locus of a brilliant point may be a curve, a surface or a region of more

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* Presented to the American Mathematical Society at its meeting, 22 February, 1902. A communication on this subject has been made to The Academy of Science of St. Louis; *cf. Trans. Acad. Sci. of St. Louis*, vol. x, No 11 (1900), p. lxii.

† Alhazen (A. D. 987–1038) determined geometrically the brilliant points of a concave mirror when the source and the recipient are given. *Cf.* Ball, *A Short History of Mathematics*, p. 167.

dimensions. When it is a curve, it is called a *brilliant curve*, and when a surface, a *brilliant surface*. Owing to the fact that the impression of an image on the retina of the eye remains for some time after the object which has produced it has disappeared or become displaced, it is possible for us to perceive the brilliant curve when the illuminated polished surface is rapidly moved.

A striking example of this fact is presented by an illuminated polished rod which is rapidly rotated. When the rod is of small radius and the axis of rotation intersects and is perpendicular to the rod, the brilliant curve lies in a plane. Fig. 3 illustrates this case. The plane of the paper represents the plane of rotation, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. This curve may be seen in a carriage wheel. In this case the source of light is either the sun or a street lamp. Owing to the great distances of the source of light and the eye of the observer from the hub of the wheel (great in comparison with the diameter of the wheel), the brilliant curve looks like a straight line passing through the hub, and on account of the motion of the carriage this line continually changes its position.

Closely packed assemblages of brilliant points look like continuous regions of light when the consecutive brilliant points are so near each other that the eye can not separate them. An example of this is presented by a circular saw which has been polished with emery in a lathe. The consecutive scratches made by the particles of emery are so close that their corresponding brilliant points are too close to be separated by the eye. Each scratch may be considered as being the special position of a variable scratch, and hence the isolated brilliant points are points of a brilliant curve. This curve is illustrated by Fig. 2 and the photographs. In Fig. 2 the plane of the paper below the line AB represents the plane of the saw, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. The photographs also show this curve. In the latter case the source of light is an electric arc and the eye is replaced by the optical centre of the lens of the camera. It will be observed, in the photographs, that near the centre of the saw there is a discontinuity in the brilliant curve. This is due to the fact that, at a small distance from the centre, the concentric circular scratches are replaced by spiral scratches, due to a different mode of polishing.

The most general problem to be discussed in this paper is the following :— *Required to find the locus of the brilliant points of a two parameter family of space curves when the source of light and the eye of an observer are in given fixed positions.* An example under this case is presented by the rapid rotation of the saw above referred to, about an axis in its plane. Another example is furnished by a family of parallel polished wires, as for instance the wires strung on the arms of a telegraph pole.

In what has been said above it has been tacitly assumed that the source of light is a point and that all the rays reflected at the brilliant points belong to a family of right lines each member of which passes through the same point, namely, the pupil of the observer's eye. It might be required to find those points of reflection for which the reflected rays belong to a family of right lines each member of which is normal to a given surface. This given surface would be called the *recipient*. The eye would then be said to be a *point recipient*. A surface to which all the incident rays are normal would be called the *source*. When all the incident rays emanate from a single point, this point would be said to be a *point source*.

In the present paper I shall confine myself to the brilliant points of curves with respect to a point source and a point recipient.

2. Definition of Brilliant Points. Given the space curve

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0,$$

a family of right lines emanating from the point P_1, (x_1, y_1, z_1) called the *source*, a family of right lines emanating from the point P_2, (x_2, y_2, z_2) called the *recipient*. The point P_0, (x_0, y_0, z_0) is said to be a *brilliant point of the space curve* $F_1 = 0$, $F_2 = 0$ *with respect to the points* P_1 *and* P_2 when the following conditions are fulfilled.

1. The point P_0 is a point of the curve $F_1 = 0$, $F_2 = 0$;
2. The right line $P_0 P_1$, considered as a whole, and the right line $P_0 P_2$, considered as a whole, make equal angles with the line of intersection of their plane with the plane normal to the curve $F_1 = 0$, $F_2 = 0$ at P_0. If the two planes coincide, P_0 shall still be considered as fulfilling this condition.

This is the most natural definition of the brilliant point of a space curve as suggested by that of a surface. Condition 2 is co-extensive with the following condition.

3. The right line $P_0 P_1$, considered as a whole, and the right line $P_0 P_2$, considered as a whole, make equal angles with the right line which is tangent to the curve $F_1 = 0$, $F_2 = 0$ at P_0.

Fig. 1 enables us to see geometrically that Conditions 2 and 3 are co-extensive. In the figure, P_0 is the centre of a sphere which is pierced in the points A, A by the tangent at P_0 to the space curve $F_1 = 0$, $F_2 = 0$, in the point B_1 by the right line $P_0 P_1$, in the point B_2 by the right line $P_0 P_2$, and in the point C by the line of intersection of the normal plane and the plane of the lines $P_0 P_1$, $P_0 P_2$. The great circle through C which has A, A for poles is the intersection of the sphere by the plane which is normal to the

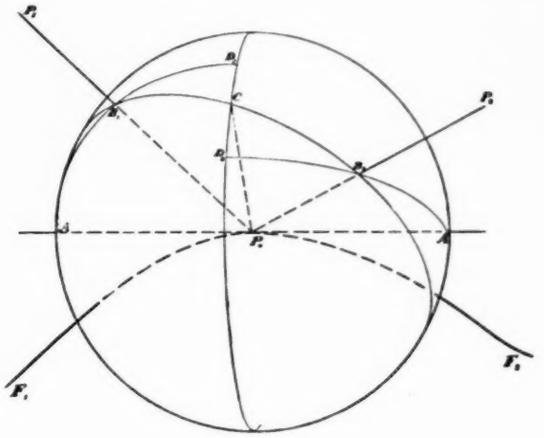

FIG. 1.

curve $F_1 = 0$, $F_2 = 0$ at P_0. The great circles AB_1 and AB_2 cut the normal plane in D_1 and D_2 respectively. In the right spherical triangles $B_1 D_1 C$ and $B_2 D_2 C$, angle $D_1 C B_1$ = angle $D_2 C B_2$. By Condition 2, side CB_1 = side CB_2. Hence the two triangles are equal and arc AB_1 = arc AB_2. But this is Condition 3. If, on the other hand, we have given arc AB_1 = arc AB_2, it follows that arc CB_1 = arc CB_2, and this is Condition 2.

Classification of Brilliant Points. Those brilliant points P_0 for which the segments of right lines $P_0 P_1$ and $P_0 P_2$ lie on opposite knappes of a cone of revolution whose axis is the tangent to the space curve at P_0 and whose vertex is P_0, shall be called *actual brilliant points*, and those for which these segments lie on the same knappe shall be called *virtual brilliant points*.

When the semi-angle of this cone is a right angle, an ambiguity arises. In this case P_0 shall be called an actual brilliant point, except when it lies between P_1 and P_2 on the right line P_1P_2. In this exceptional case, it shall be called a virtual brilliant point.

If the curve represented by the equations $F_1 = 0$, $F_2 = 0$ be the axis of a polished wire of small radius, and if one of the points P_1, P_2 be replaced by a source of light and the other by the eye of an observer, then the observer will see the actual brilliant points as luminous points.

3. Analytical Conditions. The deduction of the following conditions will be given in a paper which the author intends to publish in the near future.

The necessary and sufficient condition that the point $P,(x, y, z)$ shall be a brilliant point of the space curve $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$, with respect to the two points $P_1,(x_1, y_1, z_1)$ and $P_2,(x_2, y_2, z_2)$ is*

$$\left.\begin{aligned} F_1(x, y, z) &= 0 \\ F_2(x, y, z) &= 0 \end{aligned}\right\} \qquad (a)$$

$$\begin{aligned} &A_2\left\{B_1\left[\frac{\partial F_1}{\partial x}(x - x_2) + \frac{\partial F_1}{\partial y}(y - y_2) + \frac{\partial F_1}{\partial z}(z - z_2)\right]\right. \\ &\quad \left. + B_2\left[\frac{\partial F_1}{\partial x}(x - x_1) + \frac{\partial F_1}{\partial y}(y - y_1) + \frac{\partial F_1}{\partial z}(z - z_1)\right]\right\} \\ &- A_1\left\{B_1\left[\frac{\partial F_2}{\partial x}(x - x_2) + \frac{\partial F_2}{\partial y}(y - y_2) + \frac{\partial F_2}{\partial z}(z - z_2)\right]\right. \\ &\quad \left. + B_2\left[\frac{\partial F_2}{\partial x}(x - x_1) + \frac{\partial F_2}{\partial y}(y - y_1) + \frac{\partial F_2}{\partial z}(z - z_1)\right]\right\} \\ &\qquad = 0, \end{aligned} \qquad (b)$$

* This condition is sufficient, provided that there are no points at which the surfaces $F_1 = 0$, $F_2 = 0$ are tangent and hence

$$\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial x} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = 0;$$

and no points at which either $\frac{\partial F_1}{\partial x} = 0$, $\frac{\partial F_1}{\partial y} = 0$, $\frac{\partial F_1}{\partial z} = 0$ or $\frac{\partial F_2}{\partial x} = 0$, $\frac{\partial F_2}{\partial y} = 0$, $\frac{\partial F_2}{\partial z} = 0$.

where

$$A_i = \begin{vmatrix} \frac{\partial F_i}{\partial x} & \frac{\partial F_i}{\partial y} & \frac{\partial F_i}{\partial z} \\ x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \end{vmatrix}, \qquad B_i = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ x-x_i & y-y_i & z-z_i \end{vmatrix}, \quad i = 1, 2.$$

By putting, in equations (a), (b),

$$F_1(x, y, z) = F(x, y) = 0, \qquad F_2(x, y, z) = z = 0,$$

and hence

$$\frac{\partial F_2}{\partial x} = 0, \qquad \frac{\partial F_2}{\partial y} = 0, \qquad \frac{\partial F_2}{\partial z} = 1,$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial F}{\partial x}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F}{\partial y}, \qquad \frac{\partial F_1}{\partial z} = 0,$$

in which $F(x, y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are functions of the variables x and y only, we obtain the following theorem:

The necessary and sufficient condition that the point $P,(x, y)$ shall be a brilliant point of the plane curve $F(x, y) = 0$, with respect to the two points $P_1,(x_1, y_1, z_1)$ and $P_2,(x_2, y_2, z_2)$, which are not in the plane of the curve is

$$F(x, y) = 0 \tag{a'}$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \left\{ \left[\frac{\partial F}{\partial x}(x-x_1) + \frac{\partial F}{\partial y}(y-y_1) \right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x-x_2 & y-y_2 \end{vmatrix} \right.$$

$$\left. + \left[\frac{\partial F}{\partial x}(x-x_2) + \frac{\partial F}{\partial y}(y-y_2) \right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x-x_1 & y-y_1 \end{vmatrix} \right\}$$

$$+ z_1^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x-x_2 & y-y_2 \end{vmatrix}^2 - z_2^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x-x_1 & y-y_1 \end{vmatrix}^2 = 0 \tag{b'}$$

Let us denote by a_1 and a_2 the absolute values of z_1 and z_2 respectively

and by P_1' and P_2' the projections of the points P_1 and P_2 on the X, Y plane. The condition just stated is the same for the four cases: $(z_1 = a_1, z_2 = a_2)$, $(z_1 = -a_1, z_2 = -a_2)$. $(z_1 = -a_1, z_2 = a_2)$, $(z_1 = a_1, z_2 = -a_2)$. According as we have one of the second two, or one of the first two of these cases, the right line $P_1 P_2$ cuts the right line $P_1' P_2'$ between or not between P_1' and P_2'. Let us denote by P_i the point between P_1' and P_2' and by P_j the other point. The four points P_j, P_1', P_i, P_2' form a harmonic range. The coordinates of P_i and P_j are respectively

$$\left(\frac{x_1 + \frac{a_1}{a_2} x_2}{1 + \frac{a_1}{a_2}}, \quad \frac{y_1 + \frac{a_1}{a_2} y_2}{1 + \frac{a_1}{a_2}} \right), \quad \left(\frac{x_1 - \frac{a_1}{a_2} x_2}{1 - \frac{a_1}{a_2}}, \quad \frac{y_1 - \frac{a_1}{a_2} y_2}{1 - \frac{a_1}{a_2}} \right).$$

Both P_i and P_j satisfy equation (b'), and hence if the curve (a') passes through them, they are brilliant points.

By putting, in equation (b'), $z_1 = 0$ and $z_2 = 0$, we obtain the following theorem:

The necessary and sufficient condition that the point $P,(x, y)$ shall be a brilliant point of the plane curve $F(x, y) = 0$, with respect to the two points $P_1,(x_1, y_1)$ and $P_2,(x_2, y_2)$, which are in the plane of the curve is

$$F(x, y) = 0 \qquad (a'')$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \left\{ \left[\frac{\partial F}{\partial x}(x - x_1) + \frac{\partial F}{\partial y}(y - y_1) \right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix} \right.$$

$$\left. + \left[\frac{\partial F}{\partial x}(x - x_2) + \frac{\partial F}{\partial y}(y - y_2) \right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix} \right\} = 0 \qquad (b'')$$

The first factor of the left hand member of equation (b'') vanishes for all points of the right line $P_1 P_2$. With the exception of P_1 and P_2, the second factor vanishes for *no* point of this right line, except when the line is normal or tangent to the curve $F = 0$. To show this, substitute for x and y the coordinates of any point of the right line $P_1 P_2$

$$x = \frac{x_1 + k x_2}{1 + k}, \qquad y = \frac{y_1 + k y_2}{1 + k}.$$

The second factor of the left hand member of (b'') then becomes

$$-\frac{2k}{(1+k)^2}\left[\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)\right]\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ x_1-x_2 & y_1-y_2\end{vmatrix}.$$

This expression vanishes when $k=0$ and when $k=\infty$. These are the coordinates of P_1 and P_2. For other values of k the expression cannot vanish unless

either
$$\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)=0 \qquad (1)$$

or
$$\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ x_1-x_2 & y_1-y_2\end{vmatrix}=0. \qquad (2)$$

If the right line P_1P_2 and the curve $F(x,y)=0$ meet, equation (1) is the necessary and sufficient condition that P_1P_2 shall be tangent to $F=0$, and equation (2) is the necessary and sufficient condition that P_1P_2 shall be normal to $F=0$.

4. Loci of Brilliant Points. Suppose we have a family of space curves given by the equations

$$\Phi_1(x,y,z,p_1)=0, \qquad \Phi_2(x,y,z,p_2)=0. \qquad (a)$$

in which p_1 and p_2 are two independent parameters. If, after substituting Φ_1 and Φ_2 for F_1 and F_2 in equations (a), (b), we eliminate p_1 and p_2 between these three equations, there results an equation, independent of p_1 and p_2, which is satisfied by the brilliant points of every member of the two parameter family of space curves (a). This equation represents the surface which is the locus of all the brilliant points, *i. e.* the *brilliant surface*, of the given family of space curves with respect to the fixed points P_1 and P_2. When the functions Φ_1 and Φ_2 are of the form

$$\Phi_1(x,y,z,p_1)=F_1(x,y,z)-p_1, \qquad \Phi_2(x,y,z,p_2)=F_2(x,y,z)-p_2,$$

no elimination is necessary, and equation (b), as it stands, represents the brilliant surface. Similarly, if we have a family of plane curves given by the equation

$$\Phi(x,y,p)=0, \qquad (a')$$

and if, after substituting Φ for F in equations (a'), (b'), we eliminate p between these equations, we obtain the equation of the *brilliant curve* of the given family of plane curves (a') with respect to the fixed points P_1 and P_2, which are not in the plane of the curves. When the function Φ is of the form

$$\Phi(x, y, p) = F(x, y) - p,$$

no elimination is necessary, and equation (b'), as it stands, represents the brilliant curve. When both P_1 and P_2 lie in the plane of the family of curves (a'), we obtain the equation of the brilliant curve by eliminating p between equations (a''), (b'') after substituting Φ for F. In this case the brilliant curve consists of two distinct curves, one of which is the right line connecting P_1 and P_2.

THEOREM. *When the source and the recipient are in the plane of a family of curves, the brilliant curve of this family and that of its orthogonal trajectories are identical. The portion of the brilliant curve, with the exception of the right line $P_1 P_2$, which for the one family is the locus of the actual brilliant points, is for the other family the locus of the virtual brilliant points, and vice versa.*

5. Applications. (*a*) *The Saw Curve.* Let it be required to find the equation of the "saw curve" referred to in the introduction. The curve of light seen by an observer and shown in the accompanying photographs is that portion of the brilliant curve which is the locus of the actual brilliant points. In Fig. 2 the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The scratches caused by the particles of emery are concentric circles whose common centre is the centre of the saw, and hence our family of curves is represented by the equation

$$x^2 + y^2 - a^2 = 0, \tag{3}$$

in which a is the variable parameter. $P_1, (x_1, y_1, z_1)$ is the source and P_2, (x_2, y_2, z_2) is the recipient. The derivatives of the left hand member of equation (3) are

$$\frac{\partial F}{\partial x} = 2x, \qquad \frac{\partial F}{\partial y} = 2y.$$

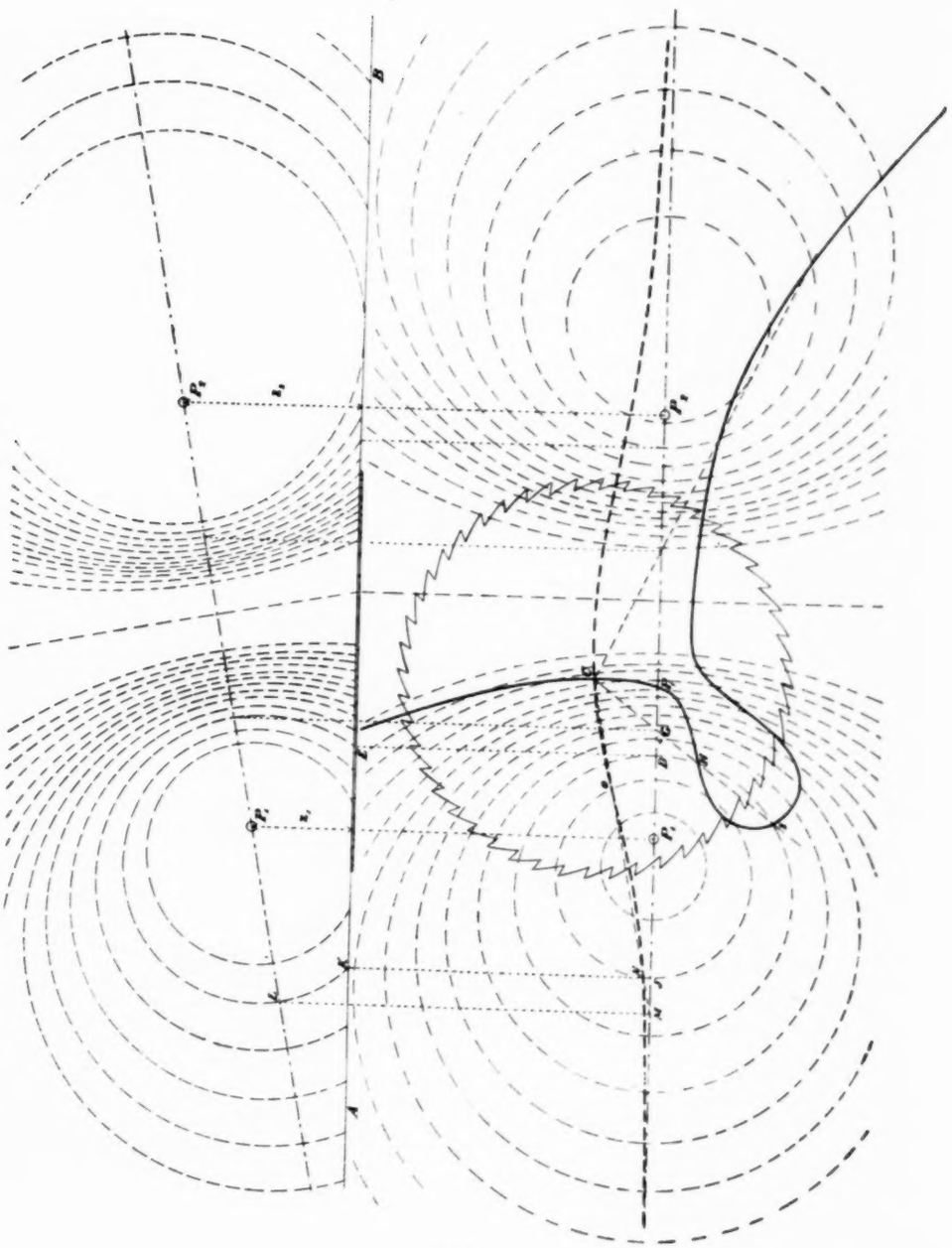

FIG. 2.

Substituting these in (b') we obtain

$$\Big[(x_2 y - y_2 x) + (y_1 x - x_1 y) + (x_1 y_2 - x_2 y_1)\Big]\Big[\{x(x-x_1) + y(y-y_1)\}$$
$$\{x_2 y - y_2 x\} + \{x(x-x_2) + y(y-y_2)\}\{x_1 y - y_1 x\}\Big]$$
$$+ z_1^2(x_2 y - y_2 x)^2 - z_2^2(x_1 y - y_1 x)^2 = 0. \qquad (4)$$

This equation may also be written as follows:

$$\Big[(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1\Big]\Big[(x_2 + x_1)(y^3 + x^2 y)$$
$$- (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy\Big]$$
$$- 2(z_1^2 x_2 y_2 - z_2^2 x_1 y_1)xy + (z_1^2 x_2^2 - z_2^2 x_1^2)y^2 + (z_1^2 y_2^2 - z_2^2 y_1^2)x^2 = 0. \qquad (5)$$

The result may be stated thus: *The brilliant curve of a family of concentric circles, when the source and recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the common centre of the circles (3), and also through the points P_i and P_j, whose coordinates have been given in §3.

When the right line $P_1' P_2'$, which is the projection of $P_1 P_2$ on the plane of the circles, passes through the centre of the circles, $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. For this relation, equation (4) becomes

$$(cy - x)^2\Big[(y_2^2 - y_1^2)(x^2 + y^2) - (y_2 - y_1)(x_1 y_2 + x_2 y_1)x - 2(y_1 y_2)(y_2 - y_1)y$$
$$+ z_1^2 y_2^2 - z_2^2 y_1^2\Big] = 0, \qquad (6)$$

in which $c = \frac{x_1}{y_1} = \frac{x_2}{y_2}$. The first factor of equation (6), when set equal to zero, represents two coincident right lines which pass through the origin and contain P_1' and P_2'. The second factor, when set equal to zero, represents the circle cut by the (X, Y)-plane from the sphere whose equation is

$$\frac{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} = k^2, \qquad (7)$$

in which $k = \frac{x_1}{x_2} = \frac{y_1}{y_2}$. This sphere cuts the right line $P_1 P_2$ in points whose

coordinates are

$$\frac{x_1 \pm k x_2}{1 \pm k}, \qquad \frac{y_1 \pm k y_2}{1 \pm k}, \qquad \frac{z_1 \pm k z_2}{1 \pm k}.$$

On account of the value of k, the first two coordinates of one of these points are zero. The result of putting $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$ can be expressed as follows:

When the axis of the saw (i. e. the right line perpendicular to the plane of the saw at its centre) intersects the right line $P_1 P_2$, which connects the source of light, P_1, and the recipient, an observer's eye, P_2, the brilliant curve consists of a straight line and a circle. The straight line passes through the centre of the saw and contains the projections of P_1 and P_2. The circle is the intersection of the plane of the saw by the sphere which has P_1 and P_2 as conjugate points, and passes through the intersection of the axis of the saw and the right line $P_1 P_2$.

When the axis of the saw intersects the right line $P_1 P_2$ in the point P_3 half way between P_1 and P_2, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at P_3, and in this case the brilliant curve consists of two right lines which are perpendicular to each other and one of which passes through the centre of the saw. When in this special case the right line $P_1 P_2$ is parallel to the plane of the saw, the two right lines cross each other perpendicularly at the centre of the saw.

The photographs show some of the special cases.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (5) degenerates into the curves represented by the equations*

$$(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1 = 0 \tag{8}$$

and

$$(x_2 + x_1)(y^3 + x^2 y) - (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy = 0. \tag{9}$$

Equations (8), (9) might have been gotten directly from (b''). It is, of course, a physical impossibility to see the curve represented by equation (9).

(b) *The Carriage Wheel Curve.* Let it be required to find the equation of the "carriage wheel curve" referred to in the introduction. In Fig. 3

* Equation (9) is the equation obtained by Lieutenant Hamilton, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 97. A geometrical construction for this curve is given by Eagles, *Constructive Geometry of Plane Curves*, p. 333.

the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The spokes of the wheel being radial and of small radius, our family of curves is represented by the equation

$$y - ax = 0. \tag{10}$$

$$\therefore \qquad \frac{\partial F}{\partial x} = -a, \qquad \frac{\partial F}{\partial y} = 1.$$

Substituting these values in equation (b'), and eliminating a between the equation thus obtained and equation (10), we get

$$\Big[(x_2y - y_2x) + (y_1x - x_1y) + (x_1y_2 - x_2y_1)\Big](-1)\Big[\{x_1y - y_1x\}\{x(x - x_2) + y(y - y_2)\} + \{x_2y - y_2x\}\{x(x - x_1) + y(y - y_1)\}\Big] +$$
$$z_1^2\{x(x - x_2) + y(y - y_2)\}^2 - z_2^2\{x(x - x_1) + y(y - y_1)\}^2 = 0. \tag{11}$$

The result may be stated as follows: *The brilliant curve of a family of radiating right lines, when the source and the recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the *radiant*, and also through the points P_i and P_j, whose coordinates have been given in §3. The first term of equation (11) differs from that of equation (4) only in sign.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (11) degenerates into the curves represented by equations (8), (9). This is in accordance with the theorem of §4, since the families of curves represented by equations (3) and (10) are orthogonal trajectories.

(*c*) *A Family of Equilateral Hyperbolas.* Let it be required to find the brilliant curve of the family of plane curves

$$xy = a, \tag{12}$$

when the source and recipient are not in the plane of the curves.

$$\frac{\partial F}{\partial x} = y, \qquad \frac{\partial F}{\partial y} = x.$$

Substituting these values in (b'), we get

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \end{vmatrix} \Big[\{y(x-x_1) + x(y-y_1)\}\{y(y-y_2) - x(x-x_2)\} + \{y(x-x_2)$$

$$+ x(y-y_2)\}\{y(y-y_1) - x(x-x_1)\} \Big] + z_1^2\{y(y-y_2) - x(x-x_2)\}^2$$

$$- z_2^2\{y(y-y_1) - x(x-x_1)\}^2 = 0 \qquad (13)$$

6. Geometrical Constructions of the Brilliant Curve for Certain Families of Plane Curves. It is a property of the sphere that, if any point of its surface be connected by right lines with a pair of conjugate points and also with the points in which the sphere is pierced by the diameter containing the conjugate points, the latter lines bisect internally and externally the angle formed by the former.

We can make use of this property to test whether a point P of a space curve is a brilliant point with respect to a source P_1 and a recipient P_2. At P draw a plane normal to the curve and through the point P_a in which this plane is pierced by the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. The point P is or is not a brilliant point according as this sphere does or does not pass through it. If the normal plane is parallel to the right line $P_1 P_2$, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at the point half way between P_1 and P_2. If the normal plane contains both P_1 and P_2, the point P is by definition a brilliant point. On this principle is based a geometrical construction for finding the brilliant curve of a family of parallel curves, whose common plane we shall call the horizontal plane, with respect to a source and a recipient which are not in the horizontal plane:

To any curve, at any point, draw a normal; this line will be normal to every curve of the family. At the point in which it cuts the horizontal projection of $P_1 P_2$ draw a vertical line, and through the point in which this line cuts the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. This sphere cuts the horizontal plane in a circle which cuts the normal in two points of the required locus.

In Fig. 2 this method is used for finding the brilliant curve of a family of concentric circles.

From the second definition of a brilliant point (see Condition 3) we

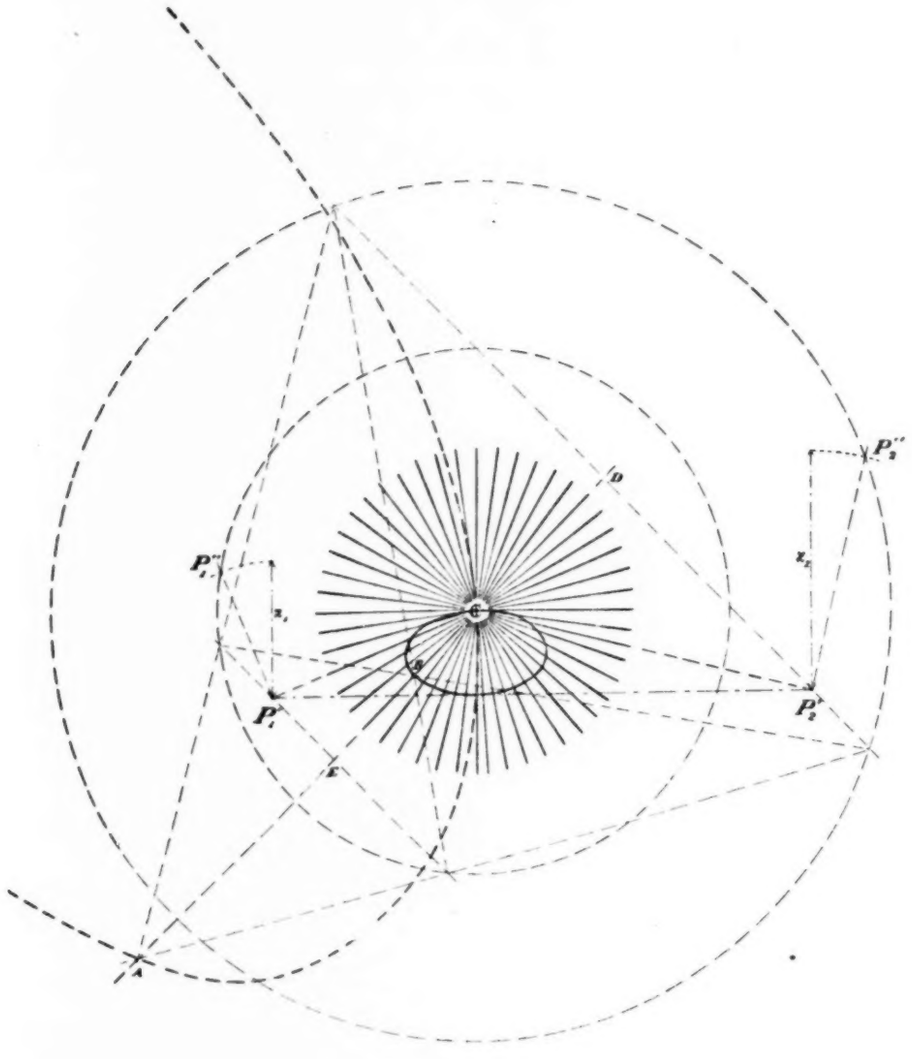

FIG. 3.

obtain the following geometrical construction for finding the brilliant points of a right line l with respect to a point source P_1 and a point recipient P_2.

About l as an axis revolve P_1 until it comes into the plane of l and P_2. There will be two revolved positions. Connect by right lines P_2 with each of the revolved positions of P_1. These right lines cut the line l in its brilliant points.

To find the locus of the brilliant points of a family of right lines, find the brilliant points of each member of the family separately.

In Fig. 3 this method is used for finding the brilliant curve of a family of radiating right lines, all of which lie in one plane, with respect to a source and recipient not in this plane.

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PROBLEMS IN INFINITE SERIES AND DEFINITE INTEGRALS; WITH A STATEMENT OF CERTAIN SUFFICIENT CONDITIONS WHICH ARE FUNDAMENTAL IN THE THEORY OF DEFINITE INTEGRALS.

By W. F. Osgood.

Simple sufficient conditions that a convergent series of continuous real functions of a real variable (or variables)

$$u_1(x) + u_2(x) + \cdots$$

may represent a continuous function and may be integrated or differentiated term by term are now pretty generally known;—I refer to the conditions stated at the beginning of §1. That equally simple conditions, closely analogous to these, exist for the *Definite Integral*

$$\int_a^\infty f(x, \alpha)\, dx$$

is less generally known and it is believed that a statement of these conditions will be welcome to the readers of the Annals.* Here α shall lie in the interval (a, b) and $f(x, \alpha)$ shall be a continuous function of the two independent real variables x, α for all values of these variables belonging to the domain

$$S: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x. \end{cases}$$

It is assumed that the integral converges for every value of α in the interval (a, b); *i. e.* that, α being given any such value and then held fast, the integral

$$\int_g^x f(x, \alpha)\, dx$$

* These conditions were obtained by Ch.-J. de la Vallée-Poussin in a noted paper: Étude des intégrales à limites infinies pour lesquelles la fonction sous le signe est continue, *Annales de la Société scientifique de Bruxelles*, vol. 16 (1891/92), p. 150; continued in *Journal de mathématiques*, ser. 4, vol. 8 (1892), p. 421. *Cf.* also C. Jordan, *Cours d'analyse*, vol. 2, 2d ed., §§64–72; O. Stolz, *Diff.- u. Integralrechnung*, vol. 3, chap. 16.

It is to be regretted that the late Professor Brunel in his excellent article on Definite Integrals, *Encyklopädie der mathematischen Wissenschaften*, vol. 2, p. 135, did not state these conditions. He gives but one of them, and that, one of the more elaborate for dealing with the special integral of §3 below.

will approach a limit when x becomes infinite. This limit will be a function of α; we write it as $\phi(\alpha)$. By definition, then,

$$\phi(\alpha) = \lim_{x=\infty} \int_g^x f(x, \alpha) dx = \int_g^\infty f(x, \alpha) dx.$$

The questions which present themselves are:

(a) When will the function $\phi(\alpha)$ be continuous?

(b) If continuous, when can it be integrated by integrating the integral under the sign of integration, *i. e.* when will

$$\int_{\alpha_0}^{\alpha} \phi(\alpha) d\alpha = \int_g^x dx \int_{\alpha_0}^{\alpha} f(x, \alpha) dx \quad ?$$

(c) When can $\phi(\alpha)$ be differentiated by differentiating the integral under the sign of integration; *i. e.* when will

$$\frac{d\phi(\alpha)}{d\alpha} = \int_g^x \frac{\partial f(x, \alpha)}{\partial \alpha} dx \quad ?$$

In the great majority of cases that arise in practice the answer to these questions is not difficult. It is stated below in §2.

§1.

The Tests for an Infinite Series.

Theorem A. *If each term of the series*

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of the real variable x in the interval $a \leqq x \leqq b$ and if the series converges uniformly in this interval, then the function $f(x)$ represented by the series is continuous in this interval.

The Weierstrassian M-Test. *The above series will converge uniformly if there exists a convergent series of positive constants, $\sum_{n=1}^{\infty} M_n$, such that, no matter what value x may have in the interval,*

$$|u_n(x)| \leqq M_n, \qquad n = 1, 2, \cdots$$

Theorem B. *If the above series converges uniformly, it can be integrated term by term:*

$$\int_{x_0}^{x_1} f(x) dx = \int_{x_0}^{x_1} u_1(x) dx + \int_{x_0}^{x_1} u_2(x) dx + \cdots,$$

where x_0, x_1, are any two numbers of the interval (a, b).

THEOREM C. *If each term of the convergent series*

$$f(x) = u_1(x) + u_2(x) + \cdots$$

has a derivative $u_n'(x)$ *which is continuous throughout the interval* $a \leqq x \leqq b$ *and if the series of the derivatives*

$$u_1'(x) + u_2'(x) + \cdots$$

converges uniformly throughout this interval, then $f(x)$ *has a derivative at each point of the interval and the derivative is given by the formula*:

$$f'(x) = u_1'(x) + u_2'(x) + \cdots .$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral is extended to infinity.

§2.

THE TESTS FOR A DEFINITE INTEGRAL.

Let $f(x, \alpha)$ be a continuous function of the real variables x, α at all points of the region

$$S: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x. \end{cases}$$

The integral

$$\int_g^h f(x, \alpha)\, dx,$$

h being a constant, is a *proper integral* and it defines a continuous function of α, $\phi(\alpha)$. This appears at once if we interpret the integral geometrically as the area under the curve in which the surface $z = f(x, y)$ is cut by the plane $y = \alpha$.*

If, however, the upper limit of integration is infinite, we have to do with an *improper integral*, and it by no means follows, even if this integral:

$$\int_g^\infty f(x, \alpha)\, dx$$

converges for all values of α in question, that the function which it represents, $\phi(\alpha)$, is continuous. For example, we see by direct computation that the integral

$$\int_0^\infty \alpha e^{-\alpha x}\, dx$$

* For an analytic proof *cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

converges for all values of $a \geqq 0$; but the function which it represents, $\phi(x)$, is discontinuous:

$$\phi(a) = 1, \qquad a > 0,$$
$$\phi(a) = 0, \qquad a = 0.$$

Again, the integral

$$\int_0^\infty \sin \pi a \; e^{-\sin^2 ax} \; dx,$$

obtained from the foregoing in a simple manner, converges for all values of a; but it is discontinuous for every positive or negative integral value of a, inclusive of zero.

Under what conditions, then, will the integral

$$\int_g^f f(x, a) dx,$$

assumed convergent, represent a continuous function of a? The answer to this and similar questions is given in the theorems that follow.

Definition of Uniform Convergence of an Improper Definite Integral. The integral

$$\int_g^f f(x, a) dx,$$

where $f(x, a)$ is continuous throughout the region S, is said to *converge uniformly* throughout the interval $a \leqq a \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive quantity G, independent of a, such that

$$\left| \int_a^x f(x, a) dx \right| < \epsilon, \qquad x > G.$$

THEOREM A. *If $f(x, a)$ is a function of the independent real variables x, a continuous throughout the region S, and if the integral*

$$\int_g^f f(x, a) dx$$

converges uniformly throughout the interval $a \leqq a \leqq b$, then the function $\phi(a)$ represented by the integral is continuous throughout this interval.

THE VALLÉE-POUSSIN $\mu(x)$-TEST. *The above integral will converge uniformly if there exists a function $\mu(x)$, independent of α, which satisfies the following conditions:*

$$(a) \quad \mu(x) \geqq 0, \quad x \geqq g;$$

$$(b) \quad |f(x, \alpha)| \leqq \mu(x), \quad x \geqq g;$$

$$(c) \quad \int_g^x \mu(x)\, dx \quad \textit{converges.}$$

COROLLARY. *These conditions will always be satisfied if there exists a constant $k > 1$ such that the product*

$$x^k f(x, \alpha)$$

remains finite throughout the region S.

THEOREM B. *If the above integral converges uniformly, it can be integrated under the sign of integration:*

$$\int_{\alpha_0}^{\alpha_1} \phi(\alpha) d\alpha = \int_g^x dx \int_{\alpha_0}^{\alpha_1} f(x, \alpha) d\alpha,$$

where α_0, α_1, are any two numbers of the interval (a, b).

THEOREM C. *If the above function has a first partial derivative with regard to α, $f_\alpha(x, \alpha)$, at each point of S, continuous throughout S, and if*

$$\int_g^x f_\alpha(x, \alpha) dx$$

converges uniformly throughout the interval $a \leqq \alpha \leqq b$; then the function $\phi(\alpha)$ has a derivative at each point of the interval, and

$$\frac{d\phi(\alpha)}{d\alpha} = \int_g^x f_\alpha(x, \alpha) dx.$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral with regard to α is extended to infinity.

Other Improper Integrals. In the foregoing, improper integrals were considered, one of whose limits of integration is infinite, but whose integrand is continuous throughout S. Similar theorems hold for integrals taken between finite limits, the integrand becoming infinite at one of the limits of integration.

Thus, let $f(x, a)$ be a continuous function of x, a at all points of the region

$$R: \quad \begin{cases} a \leqq a \leqq b, \\ g \leqq x < h; \end{cases}$$

but suppose that $f(x, a)$ becomes infinite in the points $x = h$, $a = a$, the integral now being

$$\int_g^h f(x, a)dx.$$

Then the definition of uniform convergence will read, *mutatis mutandis*, precisely as before except that G is now taken near to h and x lies between G and h. The four tests hold as before, with obvious modifications.

The proofs of the foregoing theorems are similar to the proofs of the corresponding theorems of §1 and serve as a useful exercise for the student who is familiar with these proofs and is just beginning the study of definite integrals from a modern standpoint.

§3.

A Special Series and a Special Integral.

I. *The Series.* Let each term of the series

$$u_1(x) + u_2(x) + \cdots$$

be continuous for all values of $x \geqq g$ and let the series converge uniformly throughout any arbitrarily assigned interval $g \leqq x \leqq h$. Then, if $f(x)$ denotes the function represented by the series, $f(x)$ is continuous and

$$\int_g^x f(x)dx = \int_g^x u_1(x)dx + \int_g^x u_2(x)dx + \cdots.$$

When can this series be integrated term by term to infinity? We must, of course, first of all require that each term be integrable to infinity; *i. e.*

1° $\int_g^\infty u_n(x)dx$ shall converge, $\qquad n = 1, 2, \ldots$

Furthermore let the above series of integrals converge uniformly throughout the *unlimited* interval $x \geqq g$; *i. e.*

2° A positive quantity ϵ being chosen arbitrarily small, it shall then be

possible to find a positive integer independent of x such that for all values of $x \geqq g$.

$$\left| \int_g^x u_{m+1}(x)dx + \cdots + \int_g^x u_{m+p}(x)dx \right| < \epsilon, \qquad p = 1, 2 \cdots$$

Then it is easy to show that the series can be integrated to infinity term by term; or, more fully,

(α) that the series of limits

$$\int_g^x u_1(x)dx + \int_g^x u_2(x)dx + \cdots$$

converges;

(β) that the integral $\int_g^x f(x)\,dx$ converges;

(γ) that

$$\int_g^x f(x)dx = \int_g^x u_1(x)dx + \int_g^x u_2(x)dx + \cdots$$

II. *The Integral.* A closely analogous sufficient condition holds for the reversal of the order of integration in the integral

$$\int_a^x d\alpha \int_g^\infty f(x, \alpha)\,dx,$$

where $f(x, \alpha)$ is a continuous function at all points (x, α) of the region

$$\bar{S}: \quad a \leqq \alpha, \quad g \leqq x.$$

1° *The integrals*

$$\int_g^x f(x, \alpha)dx, \qquad \int_a^x f(x, \alpha)d\alpha$$

shall respectively converge uniformly throughout every FIXED *interval*:

$$a \leqq \alpha \leqq b, \qquad g \leqq x \leqq h;$$

2° *The integral*

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha)dx$$

shall converge uniformly throughout the UNLIMITED *interval* $\alpha \geqq a$; *i. e. a positive quantity* ϵ *being chosen arbitrarily small, it shall then be possible to find a positive quantity* G, *independent of* α, *such that for all values of* $\alpha \geqq a$

$$\left| \int_a^\alpha d\alpha \int_G^x f(x, \alpha) dx \right| < \epsilon, \qquad x > G.$$

When these conditions are fulfilled

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha) dx = \int_g^\infty dx \int_a^\infty f(x, \alpha) d\alpha.$$

It sometimes happens that one of the integrals of Condition 1°,—the first, let us say,—does not converge uniformly in the interval (a, b), but that it converges uniformly in every interval lying inside of the region $\alpha \geqq a$, namely, $a < a' \leqq \alpha \leqq b$, where a', b are arbitrary.

In this case, if the remainder of Condition 1° and Condition 2° are satisfied, the reversal of the order of integration is still allowable.

A more elaborate test for a case of importance in practice is also given by Vallée-Poussin and is stated in the *Encyklopädie*, *l. c.*

§4.

Problems in Infinite Series and Products.

A.—Convergence.

1. Is the infinite series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdots$$

convergent or divergent?

2. Obtain by means of Cauchy's integral test for the convergence of an infinite series of positive terms (*cf.* Picard, *Traité d'analyse*, vol. 1, p. 27) an upper limit for the value of the remainder of the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \qquad p > 1.$$

Hence compute the value of this series when $p = 1\frac{1}{2}$ correct to two places of decimals. How many terms of this series would it be necessary to sum in order to obtain the same degree of approximation?

3. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n + n^n}}$$

converge? Discuss all cases.

4. Determine all values of x for which the series

$$\left(\frac{1}{3}\right)^2 x - \left(\frac{1 \cdot 2}{3 \cdot 5}\right)^2 x^2 + \left(\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)^2 x^3 \cdots$$

converges.

5. Is the series

$$\sum_{n=1}^{\infty} \frac{1}{\log (1 + n) \cdot \log (1 + n^n)}$$

convergent or divergent?

6. Show that if

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

represents a power series convergent for values of x other than $x = 0$, the series

$$a_0 + a_1 x + \frac{1}{2!} a_2 x^2 + \cdots + \frac{1}{i!} a_i x^i + \cdots$$

will converge for all values of x.

For what values of x will the series

$$\sum_{i=1}^{\infty} \frac{1}{i^2} a_i x^i$$

converge?

B.—Series of Functions.

7. Let the series

$$u_1(x) + u_2(x) + \cdots$$

be a series of functions each continuous in the interval $a \leqq x \leqq b$ and let it converge uniformly in this interval. Show that the term-by-term integral of this series:

$$\int_a^x u_1(x)\,dx + \int_a^x u_2(x)\,dx + \cdots$$

is a uniformly convergent series throughout this interval.

8. Let $s_n(m)$ be a single valued function of the two positive integers m, n, which satisfies the following conditions:

(a) $\lim_{n=\infty} s_n(m)$ exists: denote it by $f(m)$;

(b) $\lim_{m=\infty} s_n(m)$ exists: denote it by S_n;

(c) $s_n(m)$ converges uniformly when $n = \infty$:

$$| s_n(m) - s_{n'}(m) | < \epsilon, \qquad n > \nu,\ n' > \nu.$$

Then

(1) $\lim_{m=\infty} f(m)$ exists; denote it by A;

(2) $\lim_{n=\infty} S_n$ exists; denote it by B;

(3) $A = B$.

9. Apply the theorem of Problem 8 to prove that

$$\lim_{n=\infty} \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

10. From the formula

$$\cos mx = \cos^m x - \frac{m(m-1)}{1 \cdot 2} \cos^{m-2} x \sin^2 x + \cdots$$

deduce the development

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and justify each step of your work.

11. The series

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \cdots$$

converges when $x > 1$. Does the series represent a continuous function? Can it be differentiated term by term? Denoting the function represented by the series by $f(x)$, discuss rigorously the shape of the curve $y = f(x)$.

12. Let

$$a_0 + a_1 + \cdots$$

be an absolutely convergent series of constant terms, and let

$$f_0(x), \quad f_1(x), \cdots$$

be a set of functions each continuous within the interval $\alpha \leqq x \leqq \beta$ and each comprised between certain fixed limits

$$A \leqq f_i(x) \leqq B, \qquad i = 0, 1, \ldots$$

where A, B are constants. Show that the series

$$a_0 f_0(x) + a_1 f_1(x) + \cdots$$

represents a continuous function of x within this interval: $\alpha \leqq x \leqq \beta$.

13. Let $s_n(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$ and let $s_n(x)$ converge, for every rational value of x pertaining to this interval, toward a limit when n increases indefinitely. Furthermore, let

$$| s_n(x') - s_n(x) | < M | x' - x | ,$$

where M is a fixed positive quantity and x, x' are any two points of the interval. Prove that $s_n(x)$ converges uniformly toward a limit for all values of x in the interval.

14. If
$$a_0 + a_1 + \cdots$$
and
$$b_0 + b_1 + \cdots$$
are any two absolutely convergent series, show that the value of the series

$$a_0 b_0 + a_1 b_1 + \cdots$$

is given by the integral:

$$\frac{1}{2\pi} \int_0^{2\pi} f_1(e^{\theta i})\, f_2(e^{-\theta i})\, d\theta,$$

where

$$f_1(x) = \sum_0^{\infty} a_i x^i, \qquad f_2(x) = \sum_0^{\infty} b_i x^i.$$

15. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all uniformly continuous throughout the interval $a < x < b$ and if the series converges uniformly throughout this interval, show that the function $f(x)$ represented by the series approaches a limit when x approaches a.

16. Given that the series

$$u_1(x) + u_2(x) + \cdots$$

converges uniformly throughout the interval $a < x < b$ and that it converges when $x = a$. Can we infer that it converges uniformly when $a \leqq x < b$?

17. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout the interval $a < x \leqq b$, will it necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

18. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout every interval $\alpha \leqq x \leqq \beta$ lying inside of the interval (a, b): $a < \alpha < \beta < b$, will the series necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

19. Show that the series

$$\frac{1}{\sqrt{1 - 2r\mu + r^2}} = \sum_0^\infty r^n P_n(\mu), \qquad \begin{cases} -1 \leqq \mu \leqq 1, \\ -1 < r < 1, \end{cases}$$

can be differentiated term by term both as regards r and as regards μ.

Suggestion: Prove the theorem first for the value of μ in question and for a value of r numerically small; and then show that the theorem still holds when r has the value in question.

20. Given the formula

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{x^2}{2 \cdot (2n+2)} + \frac{x^4}{2 \cdot 4 \cdot (2n+2)(2n+4)} - \cdots \right\}$$

show that

$$\frac{\partial^2 J_n(x)}{\partial n \, \partial x} = \frac{\partial^2 J_n(x)}{\partial x \, \partial n}.$$

21. Show that if the series

$$V = \phi_0(x, y, z) + \phi_1(x, y, z) + \phi_2(x, y, z) + \cdots,$$

where $\phi_n(x, y, z)$ denotes a homogeneous polynomial in (x, y, z) of degree n, represents a function satisfying Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

then each function $\phi_n(x, y, z)$ must also satisfy Laplace's equation.

Suggestion: Assume that such a function V can be developed into a triple power series in x, y, z for values of the arguments numerically sufficiently restricted, the development holding for all values of the arguments in the neighborhood of the point $(0, 0, 0)$.

22. Show that if

$$u_0(x) + u_1(x) + \cdots$$

is a series of functions each continuous and having no roots in the interval $a \leqq x \leqq b$, and if

$$\left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leqq \gamma < 1, \qquad n \geqq m,$$

where γ, m, do not depend on x, then the given series is uniformly convergent within the interval.

Apply this test to the following series:

$$1 + x\alpha + \frac{x(x-1)}{2!}\alpha^2 + \frac{x(x-1)(x-2)}{3!}\alpha^3 + \cdots$$

where $$0 < \alpha < 1.$$

23. Find all the values of x for which the series

$$e^x \sin x + e^{2x} \sin 2x + e^{3x} \sin 3x + \cdots$$

converges. Does it converge uniformly for these values?

For what values of x can the series be differentiated term by term?

24. For what values of x is the series

$$1 + x + x^2 + \cdots$$

absolutely convergent? Is it uniformly convergent throughout the same interval?

25. Does the series

$$x + \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^4}{4} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^6}{6} + \cdots$$

converge uniformly in the interval $0 \leqq x \leqq 1$?

26. Show that the series

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \cdots, \qquad 0 \leqq x,$$

can be integrated term by term between any two positive finite limits.

Can the series be integrated term by term between the limits 0 and ∞ ? Can the function defined by the series be integrated between these limits?

27. Show that the series

$$2xe^{-x^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2} e^{-\frac{x^2}{(n+1)^2}} \right)$$

can be integrated term by term between any two finite limits.

Can the function defined by the series be integrated between the limits 0 and ∞ ? If so, is the value of this integral given by integrating the series term by term between these limits?

28. If each term of the series

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of x for all values of $x \geqq a$, and if the series

$$x^k u_1(x) + x^k u_2(x) + \cdots, \qquad k > 1,$$

satisfies the M-test (§ 1), then the given series may be integrated term by term from a to ∞ .

29. Show that the series

$$e^{-x^2} - \frac{x^2 e^{-x^2}}{2!} + \frac{x^4 e^{-x^2}}{4!} - \frac{x^6 e^{-x^2}}{6!} + \cdots$$

can be integrated term by term from 0 to ∞, given that $(n-1)! = n^{n-\frac{1}{2}} e^{-n} K_n$, when K_n lies between two positive constant quantities.

30. Can the function defined by the series

$$\frac{1}{(1+x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3+x)^3} + \cdots$$

be integrated from 0 to ∞ , and if so, is its value given by the term-by-term integral of the series?

C.—Double Series.

31. Show that the function

$$\log(1 + e^x)$$

can be developed into a power series in x. State carefully each theorem that

you employ.* Determine the coefficients of the first four terms in the development.

32. Develop the function

$$f(x) = \int_0^{\frac{\pi}{2}} \sin^{-1}(x \sin\phi)\, d\phi$$

into a power series in x. Justify each step. Assume the formula

$$\int_0^{\frac{\pi}{2}} \sin^n\phi\, d\phi = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-1)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot n},$$

where n is odd.

33. Develop $\sqrt[3]{\cos x}$ into a series proceeding according to ascending integral powers of x, stating accurately the theorems of which you make use. For how large a region will the series surely converge and represent the function?

Further problems in infinite series will be found in the exercises at the end of chs. 17 and 18 of Gibson's *Elementary Treatise on the Calculus*.

D. — Infinite Products.

34. Discuss the infinite product

$$(1 - \sin x)\,(1 + \sin \tfrac{1}{2}x)\,(1 - \sin \tfrac{1}{3}x) \ldots .$$

with reference to absolute and to uniform convergence.† Study the function defined by this product and plot the corresponding curve.

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* The final theorem here needed is to be found in Stolz, *Allgemeine Arithmetik*, vol. 1, ch. 10, §25. A special case of the theorem sufficiently general for the great majority of the cases that arise in practice (in particular, for the present case) is stated in my *Infinite Series*, §36.

Numerous examples of developments that can be obtained by this theorem are given in the text-books on Calculus; *cf.*, for example, Byerly, *Problems in Differential Calculus*, pp. 46–48. The analysis at the disposal of the student at that stage of his work is inadequate for complete demonstrations, and these are properly deferred for an advanced course in analysis.

† An infinite product $f_1 f_2 \ldots$ is said to converge when and only when (*a*) the factors from a definite one, f_m, on are all different from zero; (*b*) the product $f_{m+1} f_{m+2} \cdots f_{m+p}$ approaches a limit different from zero.

An infinite product $f_1(x)\, f_2(x) \ldots$ is said to converge uniformly throughout the interval $a \leqq x \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive integer m independent of x and such that

$$|\, f_{m+1}(x)\, f_{m+2}(x) \ldots f_{m+p}(x) - 1\,| < \epsilon, \qquad p = 1, 2, \ldots .$$

These definitions can, of course, be replaced by more general ones. But the products that would thus be admitted it is not useful to consider in practice.

35. Show that the infinite product

$$\cos x \cdot \cos \tfrac{1}{2}x \cdot \cos \tfrac{1}{3}x \cdot \ldots$$

converges for all values of x and discuss the function that it defines.

36. Discuss the infinite product

$$\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{3}\right)\left(1 - \frac{x}{5}\right)\left(1 + \frac{x}{7}\right) \cdot \ldots$$

with regard to absolute and to uniform convergence. Plot the function defined by the product as a curve. Find its slope when $x = 0$ and when $x = 1$.

37. Show that the above products can be developed into power series in x and determine the coefficients of the terms of degree 0, 1, 2.

§5.

Problems in Definite Integrals.

38. Show that the integral

$$\int_1^\infty x^{a-1} e^{-x}\, dx$$

converges and defines a continuous function for all values of a.

39. Show that the integral

$$\int_0^1 x^{a-1} e^{-x}\, dx$$

converges and defines a continuous function for all positive values of a.

Give the proof first by introducing as a new variable of integration $y = x^{-1}$; then by applying the theorems of §2 directly, modified for a finite domain of integration.

40. Show that the function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx$$

is continuous for all positive values of a and that

$$\Gamma'(a) = \int_0^\infty x^{a-1} \log x e^{-x} dx$$

$$\Gamma''(a) = \int_0^\infty x^{a-1} (\log x)^2 e^{-x}\, dx$$

etc.

Plot the curve $y = \Gamma(a)$, showing that it has a minimum between $a = 1$ and $a = 2$, and that it is always convex toward the a-axis.

41. The integral

$$\int_0^\infty a^3 e^{-a^2 x} dx$$

converges for all values of a and represents the function $\phi(a) = a$. Can the integral be differentiated under the sign of integration?

42. Given the formula

$$\int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi},$$

deduce the formulas

$$\int_0^\infty e^{-ax^2} dx = \tfrac{1}{2}\sqrt{\frac{\pi}{a}},$$

$$\int_0^\infty x^2 e^{-ax^2} dx = \tfrac{1}{4}\frac{\sqrt{\pi}}{a^{\frac{3}{2}}}, \text{ etc.}$$

43. Show that

$$\int_0^\infty e^{-x^2} \cos 2ax \, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-a^2}.$$

Suggestion: Denote the value of the function by u and show that

$$\frac{du}{da} + 2au = 0, \qquad u|_{a=0} = \tfrac{1}{2}\sqrt{\pi}.$$

44. Show that

$$\int_0^\infty e^{-\left(x^2 + \frac{a^2}{x^2}\right)} dx = \tfrac{1}{2}\pi e^{-2a}.$$

Is it allowable here to differentiate under the sign of integration for all values of a? If not, and if you have used the method suggested in Problem 43, have you been careful to make your solution rigorous?

45. Show 1) that the integral

$$\int_0^\infty \frac{\cos ax}{1 + x^2} dx$$

defines a continuous function, $f(a)$, for all values of a: 2) that

$$\phi(a) = \int_0^a f(a)\, da$$

is given by integration under the integral sign with respect to a; 3) that $f'(a)$ is given in general by differentiating under the integral sign with respect to a; 4) that $\phi(a)$ satisfies the differential equation:

$$\frac{d^2\phi}{da^2} = \phi - \frac{\pi}{2} \qquad \text{when } a > 0.$$

Hence determine the function $f(a)$ defined by the integral.

46. Discuss the function defined by the integral:

$$\int_0^\infty e^{-a^2x^2} \sin x \, dx.$$

47. Discuss the function

$$\phi(a) = \int_0^{\frac{\pi}{2}} \tan^a \phi \, d\phi,$$

plotting the graph and determining the essential characteristics.

48. Show that the functions

$$f(a) = \int_0^\infty e^{-x^2} \cos(ax^2) dx,$$

$$\phi(a) = \int_0^\infty e^{-x^2} \sin(ax^2) dx,$$

satisfy the linear differential equations:

$$f'(a) = \frac{-1}{2(1+a^2)} \Big[af(a) + \phi(a) \Big],$$

$$\phi'(a) = \frac{1}{2(1+a^2)} \Big[f(a) - a\phi(a) \Big].$$

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NOTE ON THE PRODUCT OF LINEAR SUBSTITUTIONS.*

By H. B. Newson.

If two linear substitutions in n variables be compounded, the product is also a linear substitution in n variables. The following method of expressing the result in determinant form is believed to be new. The proof is given for two substitutions in three variables, but the method and result are capable of immediate generalization for n variables.

Let T and T_1 be two substitutions as follows:

$$T: \quad \begin{aligned} \rho x_1 &= a_1 x + b_1 y + c_1 z, \\ \rho y_1 &= a_2 x + b_2 y + c_2 z, \\ \rho z_1 &= a_3 x + b_3 y + c_3 z, \end{aligned} \qquad T_1: \quad \begin{aligned} \rho_1 x_2 &= a_1 x_1 + \beta_1 y_1 + \gamma_1 z_1, \\ \rho_1 y_2 &= a_2 x_1 + \beta_2 y_1 + \gamma_2 z_1, \\ \rho_1 z_2 &= a_3 x_1 + \beta_3 y_1 + \gamma_3 z_1. \end{aligned}$$

The substitution T_2 is obtained by eliminating x_1, y_1, z_1 from the above equations. This may be done as follows: Find the inverse of T by solving the three equations of T for x, y, z. Thus we get

$$T^{-1}: \quad \begin{aligned} \frac{\Delta}{\rho} x &= A_1 x_1 + A_2 y_1 + A_3 z_1, \\ \frac{\Delta}{\rho} y &= B_1 x_1 + B_2 y_1 + B_3 z_1, \\ \frac{\Delta}{\rho} z &= C_1 x_1 + C_2 y_1 + C_3 z_1, \end{aligned}$$

where Δ is the determinant of T and A, B, etc. have the usual meanings.

The three equations of T^{-1} and the first one of T_1 form a system of four simultaneous linear equations: hence

$$\begin{vmatrix} -\frac{\Delta}{\rho} x & A_1 & A_2 & A_3 \\ -\frac{\Delta}{\rho} y & B_1 & B_2 & B_3 \\ -\frac{\Delta}{\rho} z & C_1 & C_2 & C_3 \\ -\rho_1 x_2 & a_1 & \beta_1 & \gamma_1 \end{vmatrix} = 0.$$

* Read before the Chicago Section of the American Mathematical Society at the Evanston meeting, 2/3 January, 1902.

This equation expresses the relation between x, y, z and x_2. Solving the last equation for x_2 we get

$$\rho\rho_1\Delta x_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}.$$

In like manner we get similar results for y_2 and z_2; thus

$$\rho\rho_1\Delta y_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}; \qquad \rho\rho_1\Delta z_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

When these three determinants are expanded, Δ divides out of both sides of the equation.

The general formula for n variables is

$$\rho\rho_1\Delta^{n-2}x_i''' = \begin{vmatrix} x_1 & A_1 & A_2 \cdots A_n \\ x_2 & B_1 & B_2 \cdots B_n \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ x_n & N_1 & N_2 \cdots N_n \\ 0 & \alpha_i & \beta_i \cdots \nu_i \end{vmatrix} \quad (i = 1 \cdots n).$$

THEOREM. *The value of x_i''' in the product of T and T_1, two linear substitutions, is proportional to the determinant formed by bordering the determinant of T^{-1}, the inverse of T, vertically by the variables of T and horizontally by the coefficients of the ith equation in T_1.*

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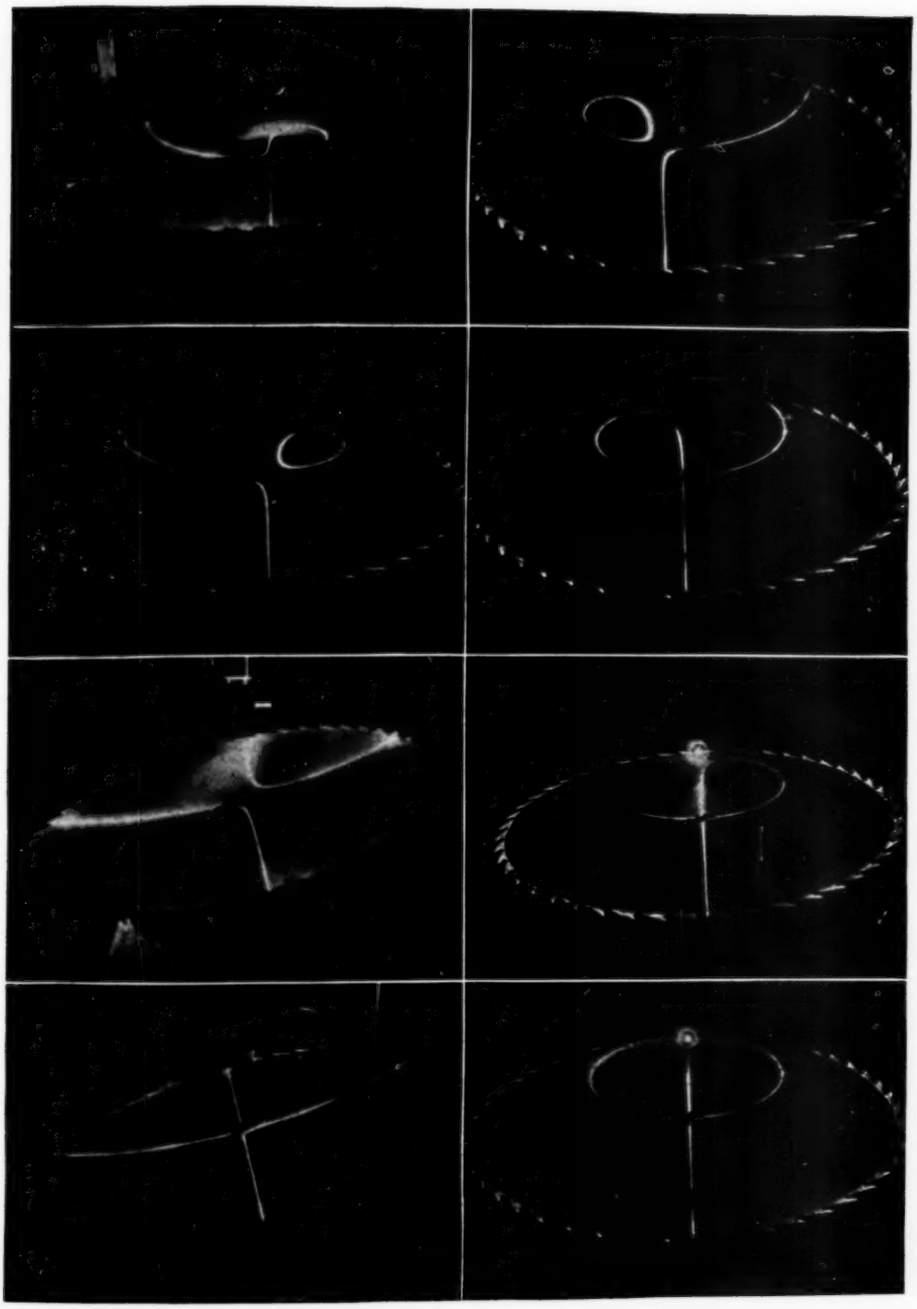

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